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A modified Halpern-type iteration algorithm for a family of hemi-relatively nonexpansive mappings and systems of equilibrium problems in Banach spaces[☆]

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ABSTRACT

In this paper, we prove strong convergence theorems by the hybrid method for a family of hemi-relatively nonexpansive mappings in a Banach space. Our results improve and extend the corresponding results given by Qin et al. [Xiaolong Qin, Yeol Je Cho, Shin Min Kang, Haiyun Zhou, Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings, Appl. Math. Lett. 22 (2009) 1051–1055], and at the same time, our iteration algorithm is different from the Kimura and Takahashi algorithm, which is a modified Mann-type iteration algorithm [Yasunori Kimura, Wataru Takahashi, On a hybrid method for a family of relatively nonexpansive mappings in Banach space, J. Math. Anal. Appl. 357 (2009) 356–363]. In addition, we succeed in applying our algorithm to systems of equilibrium problems which contain a family of equilibrium problems.

1. Introduction

Let E be a real Banach space, and let E^* be the dual space of E. Let C be a nonempty closed convex subset of E. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \to H$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(f)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality.

Equilibrium problems, which were introduced in [1] in 1994, have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity, and optimization. Numerous problems in physics, optimization, and economics reduce to finding a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [1–14] and the references therein.

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A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Iterative methods for the approximation of fixed points of a nonexpansive mapping have been studied by many researchers; see, for example, [4–6,8,15,16].

On the other hand, the hemi-relatively nonexpansive mapping, which is another generalization of a nonexpansive mapping and a relatively nonexpansive mapping, has been considered recently. Its properties and the iterative schemes for such a mapping have been studied in [12,17,18], amongst others.

Recently, Qin et al. [17] established the strong convergence of an iterative scheme with a new type of hybrid method as follows.

Theorem 1.1 (*See Qin et al. [17]*)**.** *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $T : C \to C$ be a closed and hemi-relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J T x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1. \end{cases} \tag{1.2}$$

Assume that the control sequence satisfies the restriction $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

Motivated by this result, we prove strong convergence theorems by the hybrid method for a family of hemi-relatively nonexpansive mappings in a Banach space. The main result is more general than Theorem 1.1, and at the same time, our iteration algorithm is different from the Kimura and Takahashi algorithm, which is a modified Mann-type iteration algorithm, given in [19]. In addition, we succeed in applying our algorithm to systems of equilibrium problems which contain a family of equilibrium problems.

2. Preliminaries

In what follows, E denotes a real Banach space with norm $\|\cdot\|$ and E^* the dual space of E. The norm of E^* is also denoted by$\|\cdot\|$. For $y^* \in E^*$, its value at $x \in E$ is denoted by $\langle x, y^*\rangle$.

A Banach space E is said to be strictly convex if, for $x, y \in E$, $\|x\| = \|y\| = 1$, $x \neq y$ implies that $\|x + y\| < 2$. E is said to have the Kadec–Klee property if a weakly convergent sequence $\{x_n\}$ in E with limit $x_0 \in E$ satisfies that $\lim_{n\to\infty} \|x_n\| = \|x_0\|$; then $\{x_n\}$ converges strongly to x_0. Let $S_E = \{x \in E : \|x\| = 1\}$, and define $f : S_E \times S_E \times \mathbb{R}\,\{0\} \to \mathbb{R}$ by

$$f(x, y, t) = \frac{\|x + ty\| - \|x\|}{t}$$

for $x, y \in S_E$ and $t \in \mathbb{R}\,\{0\}$. A norm of E is said to be Gâteaux differentiable if $\lim_{t\to 0} f(x, y, t)$ has a limit for each $x, y \in S_E$. In this case, E is said to be smooth. A norm of E is said to be Fréchet differentiable if $\lim_{t\to 0} f(x, y, t)$ is attained uniformly for $y \in S_E$ for each $x \in E$. It is known that E^* has a Fréchet differentiable norm if and only if E is strictly convex and reflexive, and has the Kadec–Klee property.

The normalized duality mapping of E is denoted by J; that is,

$$Jx = \{x^* \in E^* : \|x\|^2 = \langle x, x^*\rangle = \|x^*\|^2\}$$

for $x \in E$. If E is a smooth, strictly convex, and reflexive Banach space, then J is a single-valued one-to-one mapping onto E^*. In this case, the inverse mapping J^{-1} coincides with the duality mapping on E. For more details, see [20].

Suppose that a Banach space E is smooth. Then J is a single-valued mapping. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$. We know several fundamental properties of ϕ, as follows. $\phi(x, y) \geq 0$ for all $x, y \in E$. For a sequence $\{y_n\}$ in E and $x \in E$, $\{y_n\}$ is bounded if and only if $\{\phi(x, y_n)\}$ is bounded. For more details, see, for example, [5].

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong limit $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and a relatively nonexpansive mapping [21–23] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [21–23]. A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called a relatively weak nonexpansive mapping if $\widetilde{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A mapping T is called a hemi-relatively nonexpansive mapping if $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

It is obvious that a relatively nonexpansive mapping is a relatively weak nonexpansive mapping, and a relatively weak nonexpansive mapping is a hemi-relatively nonexpansive mapping; however, the converse is not true. In order to explain this better, we give the following example.

Example 2.1 (*[18]*). Let E be any smooth Banach space, and let $x_0 \neq 0$ be any element of E. We define a mapping $T : E \to E$ as follows:

$$T(x) = \begin{cases} \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0, & \text{if } x = \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0, \\ -x, & \text{if } x \neq \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0, \end{cases}$$

for $n = 1; 2; 3; \ldots$. Then T is a hemi-relatively nonexpansive mapping but not a relatively nonexpansive mapping.

Next, we give some important examples which are hemi-relatively nonexpansive.

Example 2.2 (*[10]*). Let E be a strictly convex reflexive smooth Banach space. Let A be a maximal monotone operator of E into E^* and let J_r be the resolvent for A with $r > 0$. Then $J_r = (J + rA)^{-1}J$ is a hemi-relatively nonexpansive mapping from E onto $D(A)$ with $F(J_r) = A^{-1}0$.

Remark 2.3. There are other examples of hemi-relatively nonexpansive mappings such as the generalized projections (or projections), and others; see [10].

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a reflexive Banach space E. We define two subsets $s - Li_n C_n$ and $w - Ls_n C_n$ as follows: $x \in s - Li_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and such that $x_n \subset C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in w - Ls_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and such that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. We define the Mosco convergence [24] of $\{C_n\}$ as follows. If C_0 satisfies that $C_0 = s - Li_n C_n = w - Ls_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco, and we write $C_0 = M - \lim_{n\to\infty} C_n$. For more details, see [25].

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Then, for arbitrarily fixed $x \in E$, a function $C \ni \|y - x\| \in \mathbb{R}$ has a unique minimizer $y_x \in C$. Using such a point, we define the metric projection P_C by $P_C x = y_x = \arg\min_{y\in C} \|x - y\|^2$ for every $x \in E$. In a similar fashion, we can see that a function $C \ni y \mapsto \phi(x, y) \in \mathbb{R}$ also has a unique minimizer $z_x \in C$. The generalized projection Π_C of E onto C is defined by $\Pi_C x = z_x = \arg\min_{y\in C} \phi(x, y)$ for every $x \in E$; see [1].

In order to prove our results, the following theorem plays an important role.

Theorem 2.4 (*[26]*). *Let E be a smooth, reflexive, and strictly convex Banach space having the Kadec–Klee property. Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of E. If $K_0 = M - \lim_{n\to\infty} K_n$ exists and is nonempty, then $\{\Pi_{K_n} x\}$ converges strongly to $\Pi_{K_0} x$ for each $x \in C$.*

3. A modified Halpern-type iteration algorithm for a family of hemi-relatively nonexpansive mappings

First, let us consider a family of hemi-relatively nonexpansive mappings $\{S_\lambda : \lambda \in \Lambda\}$ defined on a nonempty closed convex subset of a Banach space, and suppose that there exists a common fixed point. We prove that an iterative scheme generated by $\{S_\lambda\}$ converges strongly to a point of the set $\bigcap_\lambda F(S_\lambda)$ of common fixed points of $\{S_\lambda\}$. The iterative scheme we adopt is originally defined in [18].

The following lemma is also important in our main results.

Lemma 3.1. *Let E be a strictly convex reflexive Banach space having a Fréchet differentiable norm, C a nonempty closed convex subset of E, and $\{S_i\}$ a sequence of mappings of C into itself. Let $\{x_i\}$ be a strongly convergent sequence in C with a limit x_0 and $\{y_i\}$ a sequence in C defined by $y_i = J^{-1}(\alpha_i J x_1 + (1 - \alpha_i) J S_i x_i)$ for each $i \in \mathbb{N}$, where $\{\alpha_i\}$ is a convergent sequence in $[0, 1]$ with $\lim_{i\to\infty} \alpha_i = 0$. Suppose that $\phi(x_0, y_i) \le \phi(x_0, x_i)$ for all $i \in \mathbb{N}$ and that $\{J y_i\}$ converges weakly to $y_0^* \in E^*$. Then, $\{J x_i - J S_i x_i\}$ converges strongly to 0. Moreover, if E has the Kadec–Klee property, then $\{S_i x_i\}$ converges strongly to x_0.*

Proof. Since $\phi(x_0, y_i) \le \phi(x_0, x_i)$ for all $i \in \mathbb{N}$, we have that

$$0 \le \lim_{i\to\infty} \phi(x_0, y_i) \le \lim_{i\to\infty} \phi(x_0, x_i) = 0,$$

and hence $\lim_{i\to\infty} \phi(x_0, y_i) = 0$. Since

$$(\|x_0\| - \|y_i\|)^2 = \|x_0\|^2 - 2\|x_0\| \|y_i\| + \|y_i\|^2 \le \phi(x_0, y_i)$$

for $i \in \mathbb{N}$, we obtain that $\lim_{i\to\infty} \|y_i\| = \|x_0\|$ and that

$$\lim_{i\to\infty} \langle x_0, J y_i\rangle = \lim_{i\to\infty} \frac{1}{2}(\|x_0\|^2 + \|y_i\|^2 - \phi(x_0, y_i)) = \|x_0\|^2.$$

Using weak lower semicontinuity of the norm, we have that

$$\begin{aligned}\|x_0\|^2 &= \lim_{i\to\infty}\langle x_0, Jy_i\rangle = \langle x_0, y_0^*\rangle \le \|x_0\|\,\|y_0^*\|\\ &\le \|x_0\| \liminf_{i\to\infty}\|Jy_i\| = \|x_0\| \lim_{i\to\infty}\|Jy_i\|\\ &= \|x_0\|^2.\end{aligned}$$

Therefore, we have that $\|y_0^*\|^2 = \langle x_0, y_0^*\rangle = \|x_0\|^2$ and hence $y_0^* = Jx_0$. Thus we have that $\{Jy_i\}$ converges weakly to Jx_0. It also holds that

$$\lim_{i\to\infty}\|Jy_i\| = \lim_{i\to\infty}\|y_i\| = \|x_0\| = \|Jx_0\|.$$

Since E has a Fréchet differentiable norm, it follows that E^* has the Kadec–Klee property, and thus we have that $\{Jy_i\}$ converges strongly to Jx_0. Then, we have that

$$\begin{aligned}\|Jx_0 - Jy_i\| &= \|Jx_0 - (\alpha_i Jx_1 + (1-\alpha_i)JS_ix_i)\|\\ &\ge \|Jx_0 - \alpha_i Jx_0 - (1-\alpha_i)JS_ix_i\| - \alpha_i\|Jx_1 - Jx_0\|\\ &= (1-\alpha_i)\|Jx_0 - JS_ix_i\| - \alpha_i\|Jx_1 - Jx_0\|\end{aligned}$$

for $i \in \mathbb{N}$. Using norm-to-norm continuity of J, and $\alpha_i \to 0$, we get that

$$\lim_{i\to\infty}[(1-\alpha_i)\|Jx_0 - JS_ix_i\| - \alpha_i\|Jx_1 - Jx_0\|] = \lim_{n\to\infty}\|Jx_0 - JS_ix_i\| = 0.$$

We also have that $\{Jx_i\}$ converges strongly to Jx_0, and hence we obtain that $\{Jx_i - JS_ix_i\}$ converges strongly to 0. Further, let us suppose that E has the Kadec–Klee properly. Then, the norm of E^* is Fréchet differentiable, and therefore J^{-1} is norm-to-norm continuous. Hence, we have that

$$\lim_{i\to\infty}\|x_0 - Sx_i\| = \lim_{i\to\infty}\|J^{-1}Jx_0 - J^{-1}JSx_i\| = 0,$$

which is the desired result. □

Using this lemma, we obtain the following convergence theorem for a modified Halpern-type iteration algorithm generated by a family of hemi-relatively nonexpansive mappings.

Theorem 3.2. *Let E be a strictly convex reflexive Banach space having the Kadec–Klee property and a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and $\{S_\lambda : \lambda \in \Lambda\}$ a family of closed hemi-relatively nonexpansive mappings of C into itself having a common fixed point. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\lim_{n\to\infty}\alpha_n = 0$. For an arbitrarily chosen point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme:*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily},\\ C_1 = C,\\ y_{n,\lambda} = J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)JS_\lambda x_n),\\ C_{n+1} = \{z \in C_n : \sup_{\lambda\in\Lambda}\phi(z, y_{n,\lambda}) \le \alpha_n\phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\},\\ x_{n+1} = \Pi_{C_{n+1}}x \end{cases} \tag{3.1}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_F x$, where Π_F is the generalized projection from C onto F, and $F = \bigcap_{\lambda\in\Lambda} F(S_\lambda)$ is the set of common fixed points of $\{S_\lambda\}$.

Proof. First, we show that C_n is closed and convex. In view of the definition of ϕ, we can obtain that

$$\begin{aligned} C_{n+1} &= \{z \in C_n : \sup_{\lambda\in\Lambda}\phi(z, y_{n,\lambda}) \le \alpha_n\phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}\\ &= \bigcap_{\lambda\in\Lambda}\{z \in C_n : \phi(z, y_{n,\lambda}) \le \alpha_n\phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}\\ &= \bigcap_{\lambda\in\Lambda}\{z \in C : 2\alpha_n\langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle\\ &\quad - 2\langle z, Jy_{n,\lambda}\rangle - \alpha_n\|x_1\|^2 - (1-\alpha_n)\|x_n\|^2 + \|y_{n,\lambda}\|^2 \le 0\} \cap C_n, \end{aligned} \tag{3.2}$$

which yields that $\{C_n\}$ is closed and convex for every $n \in \mathbb{N}$.

Second, we prove that $F = \bigcap_{\lambda\in\wedge} F(S_\lambda) \subset C_n$, for all $n \in \mathbb{N}$. Let $p \in \bigcap_{\lambda\in\Lambda} F(S_\lambda)$. Then, for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$, we have that

$$\begin{aligned}\phi(p, y_{n,\lambda}) &= \phi(p, J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)JS_\lambda x_n))\\ &= \|p\|^2 - 2\langle p, \alpha_n Jx_1 + (1-\alpha_n)JS_\lambda x_n\rangle + \|\alpha_n Jx_1 + (1-\alpha_n)JS_\lambda x_n\|^2\end{aligned}$$

$$\begin{aligned}&\le \|p\|^2 - 2\alpha_n\langle p, Jx_1\rangle - 2(1-\alpha_n)\langle p, JS_\lambda x_n\rangle + \alpha_n\|x_1\|^2 + (1-\alpha_n)\|S_\lambda x_n\|^2\\ &= \alpha_n(\|p\|^2 - 2\langle p, Jx_1\rangle + \alpha_n\|x_1\|^2) + (1-\alpha_n)(\|p\|^2 - 2\langle p, JS_\lambda x_n\rangle + \|S_\lambda x_n\|^2)\\ &\le \alpha_n\phi(p, x_1) + (1-\alpha_n)\phi(p, x_n).\end{aligned} \tag{3.3}$$

Therefore, $p \in C_n$ for all $n \in \mathbb{N}$, and hence $F = \bigcap_{\lambda\in\Lambda} F(S_\lambda) \subset C_n$ for all $n \in \mathbb{N}$. Since F is nonempty, C_n is a nonempty closed convex subset of E, and thus Π_{C_n} exists for every $n \in \mathbb{N}$. Hence $\{x_n\}$ is well defined.

Third, we shall show that $\lim_{n\to\infty} x_n = x_0 = P_{C_0}x$. Since $\{C_n\}$ is a decreasing sequence of closed convex subsets of E such that $C_0 = \bigcap_{n=1}^{\infty} C_n$ is nonempty, it follows that

$$M - \lim_{n\to\infty} C_n = C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$

By Theorem 2.4, $\{x_n\} = \{P_{C_n}x\}$ converges strongly to $x_0 = P_{C_0}x$.

Fourth, we prove that $x_0 \in F = \bigcap_{\lambda\in\Lambda} F(S_\lambda)$. Since $x_0 \in C_n$ for every $n \in \mathbb{N}$, it follows that

$$\sup_{\lambda\in\Lambda} \phi(x_0, y_{n,\lambda}) \le \alpha_n\phi(x_0, x_1) + (1-\alpha_n)\phi(x_0, x_n)$$

for all $n \in \mathbb{N}$. Fix $\lambda \in \Lambda$ arbitrarily. From the assumption that $\lim_{n\to\infty} \alpha_n = 0$, we can take a subsequence $\{y_{n_i,\lambda}\}$ of $\{y_{n,\lambda}\}$ such that $\{Jy_{n_i,\lambda}\}$ converges weakly to a point $y^* \in E^*$. Then, by Lemma 3.1, we have that

$$\lim_{i\to\infty} \|x_0 - S_\lambda x_{n_i}\| = 0.$$

And from the closedness of S_λ, for any $\lambda \in \Lambda$, we have that $x_0 \in F(S_\lambda)$ for any $\lambda \in \Lambda$, and hence $x_0 \in F = \bigcap_{\lambda\in\Lambda} F(S_\lambda)$.

Finally, since $x_0 = \Pi_{C_0}x \in F$ and F is a nonempty closed convex subset of $C_0 = \bigcap_{n=1}^{\infty} C_n$, we conclude that $x_0 = \Pi_F x$, which completes the proof. □

Remark 3.3.

(1) We extend the main result of [17] from a hemi-relatively nonexpansive mapping to a general family of hemi-relatively nonexpansive mappings; furthermore, our method of proof is different from that in [17].

(2) Our iterative scheme is a modified Halpern-type scheme. In [19], the authors studied a modified Mann-type scheme; they are independent of each other.

Considering a single operator as a special case, we can obtain the following corollary directly.

Corollary 3.4. *Let E be a strictly convex reflexive Banach space having the Kadec–Klee property and a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and S a closed hemi-relatively nonexpansive mappings of C into itself having a common fixed point. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. For an arbitrarily chosen point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme:*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily},\\ C_1 = C,\\ y_n = J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)JSx_n),\\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \alpha_n\phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\},\\ x_{n+1} = \Pi_{C_{n+1}}x \end{cases} \tag{3.4}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$, where $\Pi_{F(S)}$ is the generalized projection from C onto $F(S)$.

4. A modified Halpern-type iteration algorithm for systems of equilibrium problems

In this section, we consider the problem of finding a solution of *EP*. Equilibrium problems provide a novel and unified treatment of a wide class of problems which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity, optimization, and so on.

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A_1) $f(x, x) = 0$, for all $x \in C$;
(A_2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$, for all $x, y \in C$;
(A_3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} f(tz + (1-t)x, y) \le f(x, y)$;
(A_4) for each $x \in C$, the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 4.1 ([1]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1)–(A_4). Let $r > 0$ and $x \in E$; then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \text{for all } y \in C.$$

Lemma 4.2 ([13]). *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1)–(A_4). Let $r > 0$ and $x \in E$, and define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\}$$

for all $z \in C$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*
$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$
3. *$F(T_r) = EP(f)$;*
4. *$EP(f)$ is closed and convex.*

Lemma 4.3 ([14]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1)–(A_4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

In order to use the iterative scheme in Theorem 3.2 to deal with equilibrium problems, we give the following conclusion, which has studied by some authors (see [10,27]).

Lemma 4.4. *Let C be a closed convex subset of uniformly smooth, strictly convex and reflexive Banach space E, and define the mapping T_r as Lemma 4.2; then T_r is a hemi-relatively nonexpansive mapping.*

Next, we consider the equilibrium problem for a family of bifunctions $\{f_\lambda : C \times C \to \mathbb{R},\ \lambda \in \Lambda\}$, which satisfy (A_1)–(A_4). Very recently, Colao, Acedo, and Marino have studied a similar case. The difference is that $\lambda \in \mathbb{N}$ in their case, and the case is said to be that of systems of equilibrium problems; see [28].

In the following, we will apply the iterative scheme (3.1) to systems of equilibrium problems.

Theorem 4.5. *Let E be a strictly convex reflexive Banach space having the Kadec–Klee property and a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E. Let $f_\lambda : C \times C \to \mathbb{R}, \lambda \in \Lambda$ be a family of bifunctions satisfying (A_1)–(A_4), and $F = \bigcap_{\lambda \in \Lambda} EP(f_\lambda) \neq \emptyset$. Suppose that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. For an arbitrarily chosen point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme:*

$$\begin{cases} x_1 \in C \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ f_\lambda(u_n, y) + \dfrac{1}{r}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C,\ r > 0, \\ y_{n,\lambda} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)Ju_{n,\lambda}), \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda \in \Lambda} \phi(z, y_{n,\lambda}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x \end{cases} \tag{4.1}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_F x$, where $F = \bigcap_{\lambda \in \Lambda} EP(f_\lambda)$.

Proof. In view of Lemma 4.2, we know that $u_{n,\lambda} = T_{r,\lambda} x_n$ and $F(T_{r,\lambda}) = EP(f_\lambda)$ for all $\lambda \in \Lambda$. So we can replace the iterative scheme (4.1) by the following:

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ y_{n,\lambda} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)JT_{r,\lambda} x_n), \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda \in \Lambda} \phi(z, y_{n,\lambda}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x. \end{cases} \tag{4.2}$$

From Lemma 4.4, we obtain that $T_{r,\lambda}$ must be a hemi-relatively nonexpansive mapping for $\lambda \in \Lambda$. And by Lemma 4.2 again, we learn that $T_{r,\lambda}$ is a family of continuous operators. So $T_{r,\lambda}$ must be closed. Using Theorem 3.2, we can get that $\{x_n\}$ converges strongly to $\Pi_{F_1} x$, where $F_1 = \bigcap_{\lambda \in \Lambda} T_{r,\lambda}$, and by Lemma 4.2, $F_1 = \bigcap_{\lambda \in \Lambda} EP(f_\lambda) = F$, which completes the proof. □

As a special case, we consider a single $EP(f)$, and we can directly give the following corollary.

Corollary 4.6. *Let E be a strictly convex reflexive Banach space having the Kadec–Klee property and a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$, $\lambda \in \wedge$ be a bifunction satisfying (A_1)–(A_4), and $EP(f) \neq \emptyset$. Suppose that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. For an arbitrarily chosen point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme:*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ f(u_n, y) + \dfrac{1}{r}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C,\ r > 0, \\ y_n = J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)Ju_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x \end{cases} \tag{4.3}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_{EP(f)}x$.

If we consider another case, that is, $\alpha_n \equiv 0$ in (4.1), we can apply Theorem 4.5 and obtain the following corollary directly.

Corollary 4.7. *Let E be a strictly convex reflexive Banach space having the Kadec–Klee property and a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E. Let $f_\lambda : C \times C \to \mathbb{R}$, $\lambda \in \Lambda$ be a family of bifunctions satisfying (A_1)–(A_4), and $F = \bigcap_{\lambda\in\Lambda} EP(f_\lambda) \neq \emptyset$. Suppose that $\alpha_n \equiv 0$. For an arbitrarily chosen point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme:*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ f_\lambda(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda\in\Lambda} \phi(z, u_{n,\lambda}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x \end{cases} \tag{4.4}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_F x$, where $F = \bigcap_{\lambda\in\Lambda} EP(f_\lambda)$.

Remark 4.8. Throughout the paper, the Mosco convergence plays an important role as the monotone CQ method does in others; see [12,17,18].

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